AN ELEMENTARY PROOF OF THE EXISTENCE OF A COMPETITIVE EQUILIBRIUM IN A SPECIAL CASE*

ROSA BARBOLLA AND LUIS C. CORCHÓN

I. INTRODUCTION

The proof of the existence of a competitive equilibrium has played an important role in the development of modern microeconomics. Such a proof usually involves the use of an advanced fixed point theorem: either Brouwer's or Kakutani's. Since competitive excess demand functions do not possess any structural properties beyond continuity and Walras' Law, it is sometimes argued that the proof of the existence of a competitive equilibrium and the proof of the existence of a fixed point are actually equivalent. Once such equivalence has been established, the following questions appears to be quite natural: how robust is the previous result to changes in the admissible space of economies? Or in other words: is it possible to introduce any additional assumption on preferences, technology, etc., such that the proof of the existence of an equilibrium is drastically simplified? In 1977 Joseph Greenberg proved that the introduction of an additional assumption, namely that goods are gross substitutes, simplified the proof of the existence of a competitive equilibrium. The present paper aims to contribute to such a line of research. We shall show that if the gross substitutes assumption is replaced by the assumption that good n is a weak gross substitute with the rest of goods and the assumption that the determinant of a Jacobian of excess demand functions is everywhere nonvanishing, we still have a simple proof of the existence of an equilibrium.

A motivation for our work can be found in the fact that equilibrium is a totally meaningful concept in the case in which uniqueness and stability are taken as granted. But precisely our condition on the Jacobian implies uniqueness (see Varian [1984, p. 244]). Moreover, a theorem by Hands [1981] on the stability of equilibrium requires a special case of our condition on the Jacobian (see Hands [1981, p. 209]). Therefore, the methods developed in this paper work for the class of economies for which the concept of equilibrium may be considered as totally satisfactory.

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II. The Model

The are n goods. The price of good i is denoted by p_i. Let $S = \{p \in R^n_+ | \Sigma^n_{i=1} p_i = 1\}$ be the price simplex. R^n_+ is the nonnegative orthant of R^n. Let $\overline{R} = RU\{+\infty\}$; i.e., the real line extended at $+\infty$. Let z: $S \rightarrow \overline{R}^n$ be an excess demand function. If $\tilde{p} \in \text{int } S, z(\tilde{p}) \in R^n$. We shall assume the following:

(1) $z(p)$ is continuously differentiable $\forall p \in \text{int } S$. $z_n(p)$ is continuous $\forall p_n \geqq 0$;
(2) $z(p)$ is homogeneous of degree zero;
(3) $p.\, z(p) = 0$ (Walras' Law);
(4) If $p \rightarrow \overline{p}$ with some zero component (but not the nth) then

$$\sum_{i=1}^{n-1} z_i(p)^2 \rightarrow +\infty.$$
$$p \rightarrow \overline{p}$$

Assumptions 1–4 are more or less standard. The continuity of $z_n(\overline{p})$ must be understood in an extended sense; i.e., if $z_n(\overline{p}) = +\infty$, and $p \rightarrow \overline{p}$, $z_n(p) \rightarrow +\infty$.

Notice that the boundary condition is usually stated in terms of all goods.

Let us denote by $z_{ij} = \partial z_i(\,)/\partial p_j$. $L(p)$ is an $(n - 1) \times (n - 1)$ matrix with typical component z_{ij}.

(5) Determinant $L(p) \neq 0\ \forall p \in S, p_i > 0, i = 1, 2, \ldots, n - 1$;
(6) $\Sigma_{i=1}^{n-1} p_i z_i(\,)$ is a nondecreasing function of p_n for $p_i > 0$, $i = 1, 2, \ldots, n - 1$.

A possible interpretation of (5) is the nonexistence of large income effects (see Varian [1984, p. 244]). This assumption is a generalization of the so-called Gale property (see Arrow-Hahn [1971, p. 208]) which in turn is a generalization of gross substitutes, dominant diagonal, and other properties that are used in the areas of uniqueness, comparative statics, and stability in general equilibrium. As we know from the work on the properties of excess demand functions [Sonnenschein, Debreu, Mas-Colell, and others], such an assumption is not necessarily true (see Shafer and Sonnenschein [1982]).

A sufficient condition that guarantees (6) is that

$$\frac{\partial z_i(\,)}{\partial p_n} \geqq 0, \qquad i = 1, 2, \ldots, n - 1;$$

i.e., good n is a weak gross substitute of goods $1, 2, \ldots, n-1$. Alternatively, if the elasticity of demand for good n with respect to p_n is negative and greater than or equal to one, (6) holds.

Comparing our assumptions with those of Greenberg [1977], we notice that our boundary condition is slightly stronger. However, our assumptions (5) and (6) can be regarded as generalizations of his gross substitute assumption. Finally, we shall define our equilibrium notion.

DEFINITION. A vector $p^* = (p_1^*, \ldots, p_n^*)$ is said to be an equilibrium price vector if

(a) $p^* \in S$;
(b) $z_i(p^*) \leqq 0$ for every $i = 1, \ldots, n$;
(c) if $z_i(p^*) < 0$, then $p_i^* = 0$.

And notice that $p_i^* > 0, i = 1, 2, \ldots, n-1$ (from (4) above).

III. THE THEOREM

In this section we shall state and prove our main theorem.

THEOREM. Under assumptions (1)–(6) above, there exists an equilibrium price vector.

Proof. Let $\overline{p} \in \operatorname{int} S$. Let us define

$$w = \sum_{i=1}^{n-1} z_i(\overline{p})^2.$$

Now let

$$B = \{p \in S | \sum_{i=1}^{n-1} z_i(p)^2 \leqq 2w\}.$$

The set B has the following properties: it is nonempty (since $\overline{p} \in B$), bounded (since $p \in S$), and closed (by the continuity of $z_i(\;)$ and assumption (4)). Also $\forall\ \tilde{p} \in B$, the first $n-1$ components of $\overline{p}$ must be strictly positive (by (4)). Therefore, $\Sigma_{i=1}^{n-1} z_i(p)^2$ is continuous on B.

Now consider the following problem:

$$\text{To min} \sum_{i=1}^{n-1} z_i(p)^2 \qquad p \in B.$$

A solution exists. Call it p^*. Also notice that the above minimization can be written in the following form:

$$\text{To } \min \sum_{i=1}^{n-1} z_i(p)^2$$

subject to

$$\begin{cases} p_i \leqq 0 \qquad i = 1, 2, \ldots, n \qquad (1) \\ \sum_{i=1}^{n} p_i = 1 \qquad (2) \\ \sum_{i=1}^{n-1} z_i(p)^2 \leqq 2\omega \qquad (3). \end{cases}$$

First, if $z_i(\bar{p}) = 0$, $i = 1, 2, \ldots, n - 1$, the argument is done. Now notice that, at any solution, (3) holds with strict inequality, as well as (1) for $i = 1, 2, \ldots, n - 1$. Moreover, the Lagrange multiplier associated with (2) is zero because of our assumption (2).

Therefore, the first $n - 1$ first-order conditions read

$$L(p^*).\, z = 0 \qquad [z = z_1 \ldots z_{n-1}],$$

and this implies that $z_i(p^*) = 0$ $i = 1, 2, \ldots, n - 1$. If $p_n^* \neq 0$, then $z_n(p^*) = 0$; i.e., p^* is an equilibrium. So let us assume that $p_n^* = 0$. In this case if $z_n(p^*) < 0$, an equilibrium exists, so we only have to consider the case in which either $z_n(p^*) > 0$ or $z_n(p^*) = +\infty$.

Let us denote by $p_{-n} = (p_1, p_2, \ldots, p_{n-1})$. If we take $p_n' \neq 0$ but sufficiently close to zero, we have,

$$0 > -p_n' z_n(p_{-n}^*, p_n') = \sum_{i=1}^{n-1} p_i^* z_i(p_{-n}^*, p_n') \geqq 0$$

(the first inequality is due to the continuity of $z_n(\ )$ and the last to condition (6)) which is a contradiction.

Assumptions (5) and (6) can be replaced by the following assumption: let $J(p)$ be the $n \times n$ matrix with typical component z_{ij}. Then

$$\det \begin{vmatrix} J(p) & p \\ p & 0 \end{vmatrix} \neq 0 \; \forall p \in S.$$

The difficulty with this approach is the economic interpretation of such a condition. It is true that when $z = 0$, this condition is equivalent to our assumption (5). (See Mas-Colell [1985, pp. 178–

80]). However, we have been unable to interpret such a condition when $z \neq 0$.

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